Lecture 5

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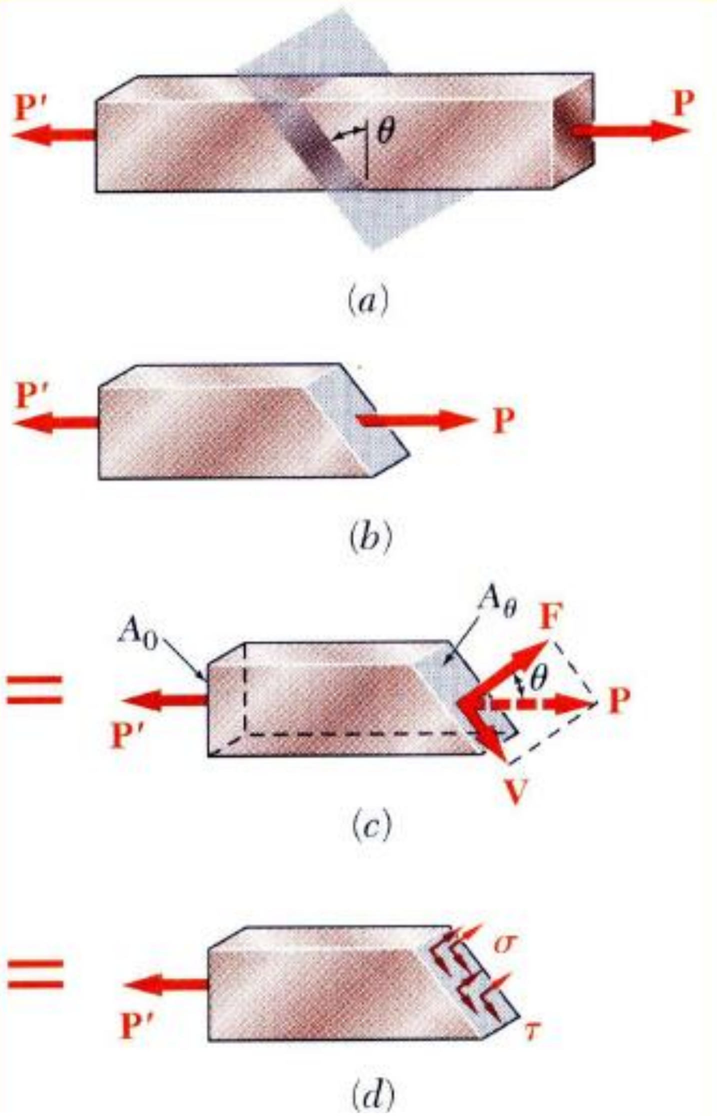
Stress on an oblique plane



- Pass a section through the member forming an angle θ with the normal plane.
- From equilibrium conditions, the distributed forces (stresses) on the plane must be equivalent to the force P.
- Resolve P into components normal and tangential to the oblique section,

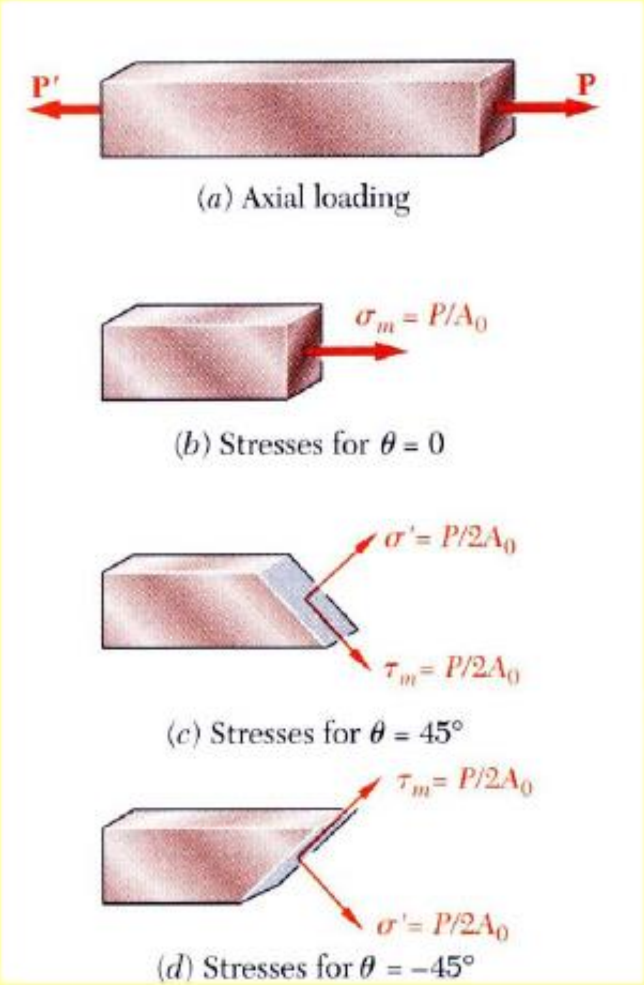
$$F = P\cos\theta \qquad V = P\sin\theta$$

- The average normal and shear stresses on the oblique plane are

$$\sigma = \frac{F}{A_\theta} = \frac{P\cos\theta}{A_0 / \cos\theta} = \frac{P}{A_0}\cos^2\theta$$

$$\tau = \frac{V}{A_\theta} = \frac{P\sin\theta}{A_0 / \cos\theta} = \frac{P}{A_0}\sin\theta\cos\theta$$

Stress on an oblique plane



- Normal and shearing stresses on an oblique plane

$$\sigma = \frac{P}{A_0}\cos^2\theta \qquad \tau = \frac{P}{A_0}\sin\theta\cos\theta$$

- The maximum normal stress occurs when the reference plane is perpendicular to the member axis,

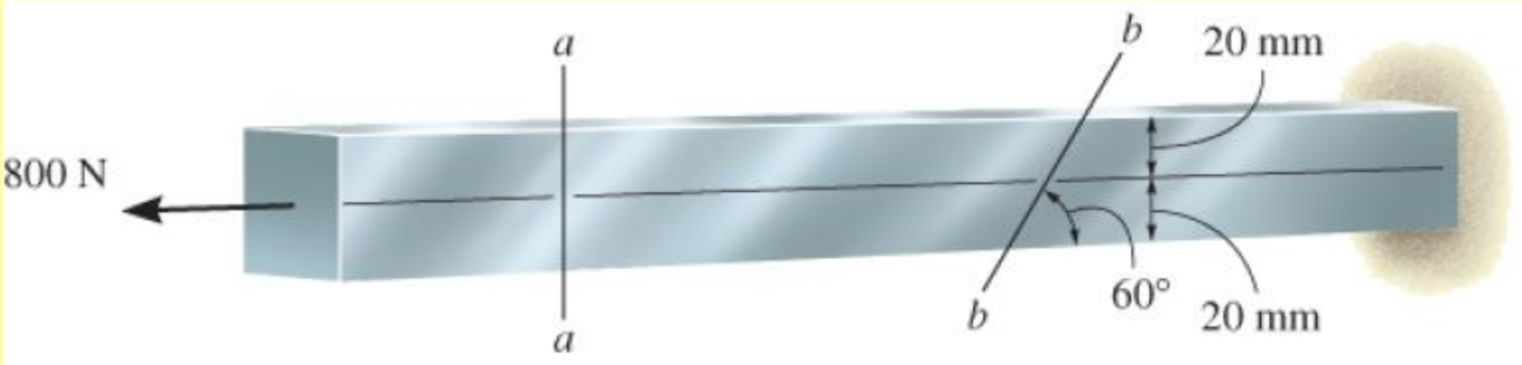
$$\sigma_m = \frac{P}{A_0} \qquad \tau' = 0$$

- The maximum shear stress occurs for a plane at $\pm$ 45° with respect to the axis,

$$\tau_m = \frac{P}{A_0}\sin 45\cos 45 = \frac{P}{2A_0} = \sigma'$$

Stress on an oblique plane

Determine average **normal stress** and average **shear stress** acting along **(a)** section planes *a-a*, and **(b)** section plane *b-b*.



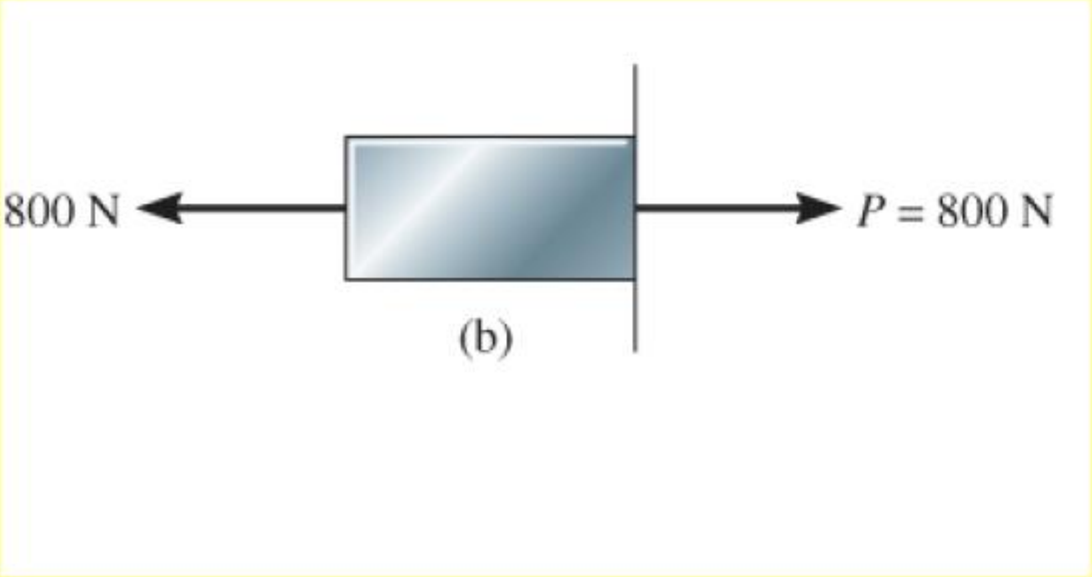
(a)

Depth and thickness = 40 mm

Stress on an oblique plane

Part (a): Internal loading

Based on free-body diagram, Resultant loading of axial force, $P = 800$ N

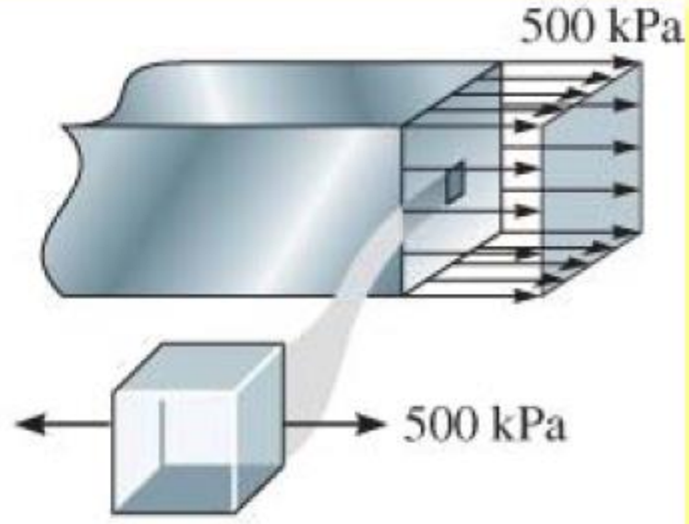


Stress on an oblique plane

Part (a): Average stress

Average normal stress, σ

$$\sigma = \frac{P}{A} = \frac{800\ \text{N}}{(0.04\ \text{m})(0.04\ \text{m})} = 500\ \text{kPa}$$

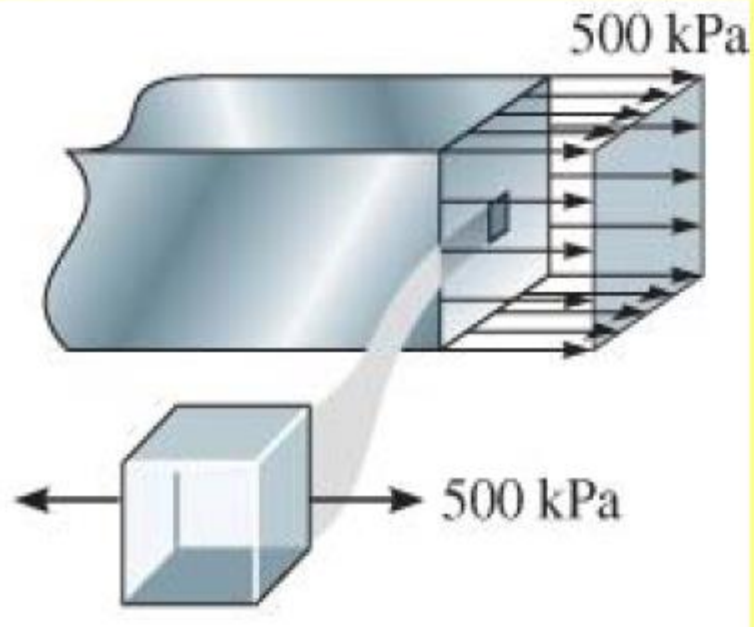


(c)

Stress on an oblique plane

Part (a): Internal loading

No shear stress on section, since shear force at section is zero.



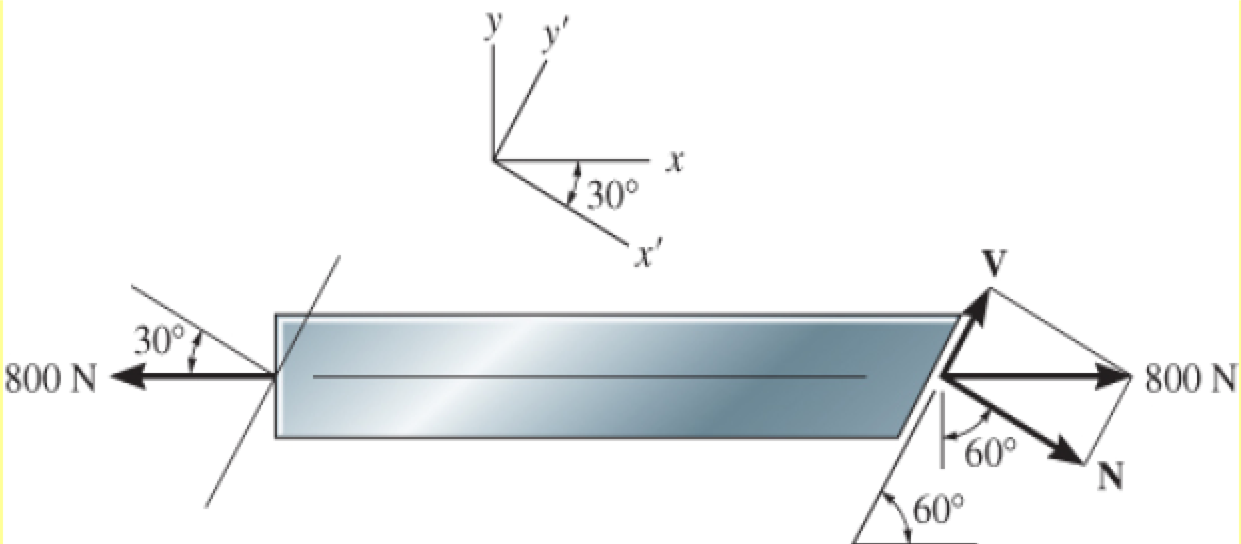
(c)

Stress on an oblique plane

Part (b): Internal loading

$$\overset{+}{\rightarrow} \sum F_x = 0; \quad -800\text{ N} + N\sin 60^\circ + V\cos 60^\circ = 0$$

$$+\uparrow \sum F_y = 0; \quad V\sin 60^\circ - N\cos 60^\circ = 0$$



(d)

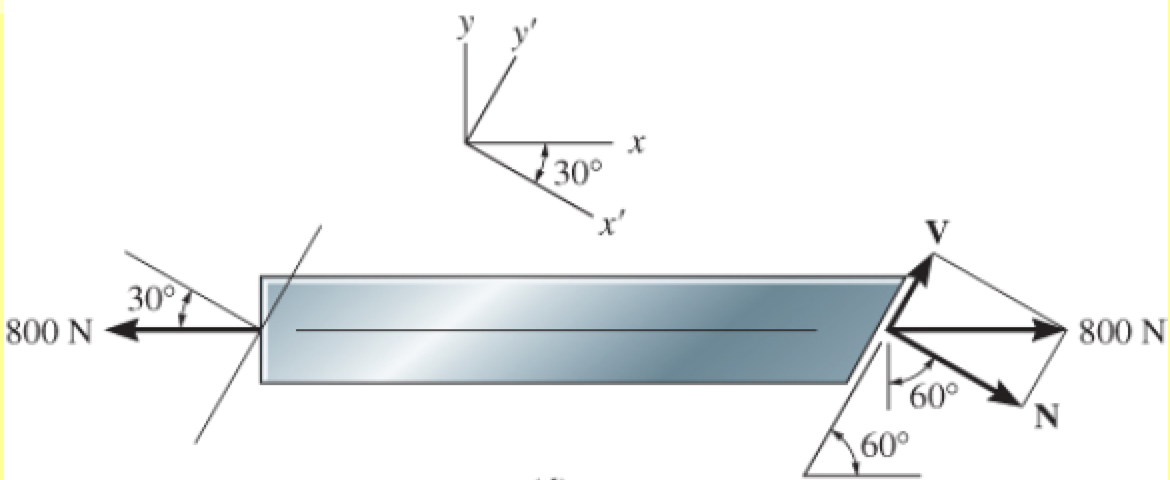
Stress on an oblique plane

Part (b): Internal loading

Or directly using x', y' axes,

$$\searrow^{+} \sum F_{x'} = 0; \qquad N - 800\text{ N}\cos 30^\circ = 0$$

$$\nearrow^{+} \sum F_{y'} = 0; \qquad V - 800\text{ N}\sin 30^\circ = 0$$

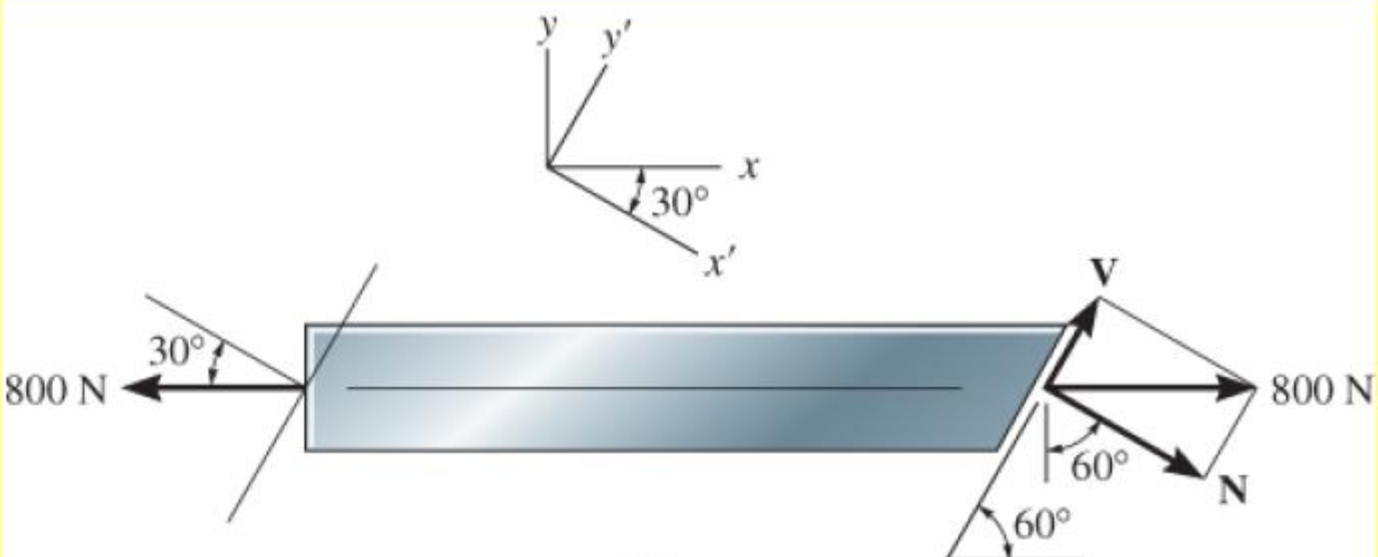


(d)

Stress on an oblique plane

Part (b) Average normal stress

$$\sigma = \frac{N}{A} = \frac{692.8\ \text{N}}{(0.04\ \text{m})(0.04\ \text{m}/\sin 60^\circ)} = 375\ \text{kPa}$$



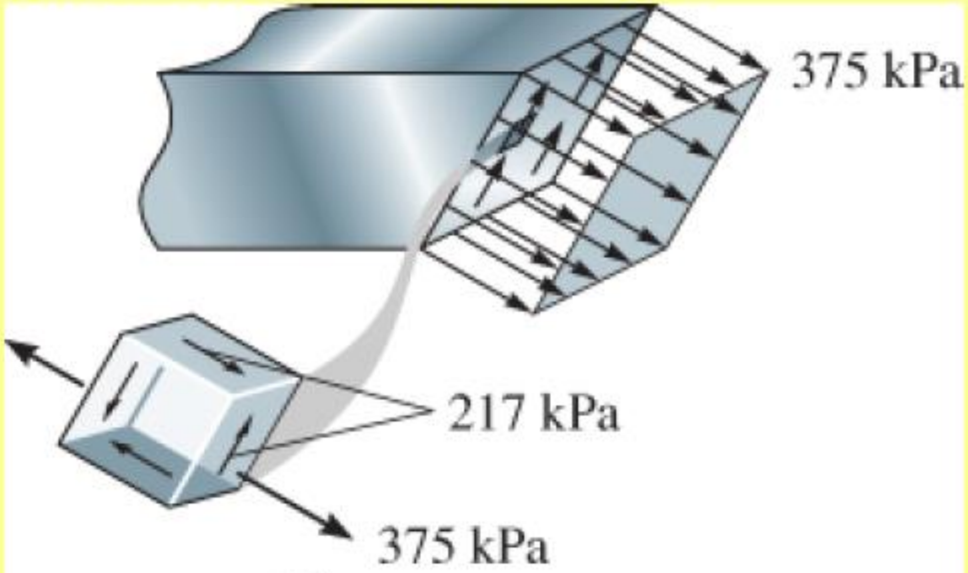
(d)

Stress on an oblique plane

Part (b): Average shear stress

$$\tau_{avg} = \frac{V}{A} = \frac{400\ \text{N}}{(0.04\ \text{m})(0.04\ \text{m}/\sin 60°)} = 217\ \text{kPa}$$

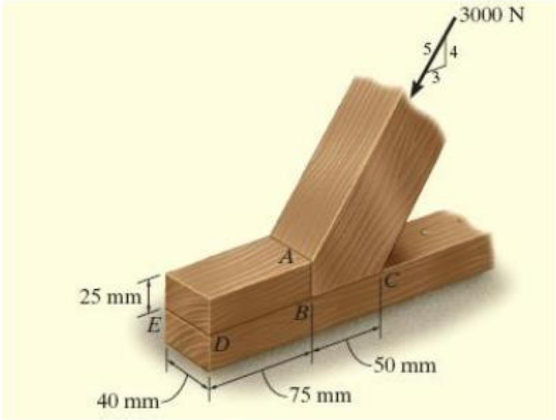
Stress distribution shown below



(e)

Example

The inclined member is subjected to a compressive force of 3000 N. Determine the average compressive stress along the smooth areas of contact defined by AB and BC, and the average shear stress along the horizontal plane defined by EDB.

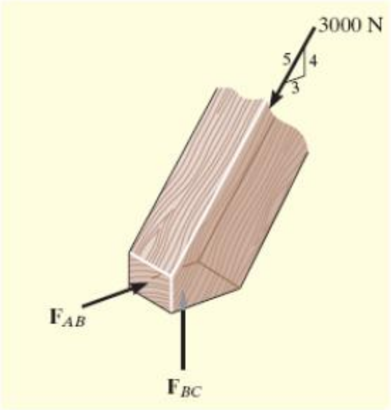


Solution:

The compressive forces acting on the areas of contact are

$$\xrightarrow{+} \sum F_x = 0; \quad F_{AB} - 3000\left(\tfrac{3}{5}\right) = 0 \Rightarrow F_{AB} = 1800 \text{ N}$$

$$+\uparrow \sum F_y = 0; \quad F_{BC} - 3000\left(\tfrac{4}{5}\right) = 0 \Rightarrow F_{BC} = 2400 \text{ N}$$

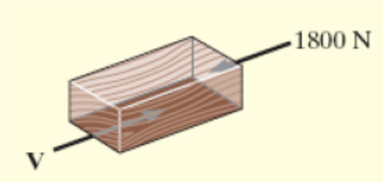


Example

Solution:

The shear force acting on the sectioned horizontal plane *EDB* is

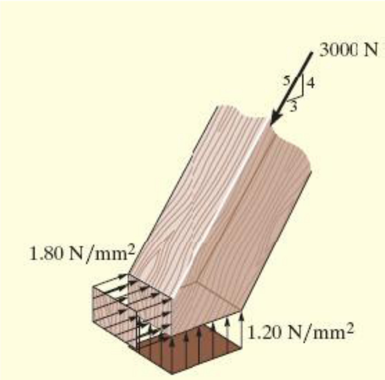
$$+\rightarrow \sum F_x = 0; \quad V = 1800\,\text{N}$$



Average compressive stresses along the AB and BC planes are

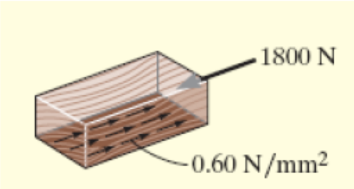
$$\sigma_{AB} = \frac{1800}{(25)(40)} = 1.80\,\text{N/mm}^2 \quad (\text{Ans})$$

$$\sigma_{BC} = \frac{2400}{(50)(40)} = 1.20\,\text{N/mm}^2 \quad (\text{Ans})$$



Average shear stress acting on the BD plane is

$$\tau_{avg} = \frac{1800}{(75)(40)} = 0.60\,\text{N/mm}^2 \quad (\text{Ans})$$



Factor of safety & allowable stress

Structural members or machines must be designed such that the working stresses are less than the ultimate strength of the material.

$$FS = \text{Factor of safety}$$

$$FS = \frac{\sigma_u}{\sigma_{all}} = \frac{\text{ultimate stress}}{\text{allowable stress}}$$

Factor of safety considerations:

- uncertainty in material properties
- uncertainty of loadings
- uncertainty of analyses
- number of loading cycles
- types of failure
- maintenance requirements and deterioration effects
- importance of member to integrity of whole structure
- risk to life and property
- influence on machine function

Factor of safety & allowable stress

- When designing a structural member or mechanical element, the stress in it must be restricted to safe level
- Choose an allowable load that is less than the load the member can fully support
- One method used is the factor of safety (F.S.)

$$\mathbf{F.S.} = \frac{F_{\text{fail}}}{F_{\text{allow}}}$$

Factor of safety & allowable stress

- If load applied is linearly related to stress developed within member, then F.S. can also be expressed as:

$$\textbf{F.S.} = \frac{\sigma_{\text{fail}}}{\sigma_{\text{allow}}} \qquad \textbf{F.S.} = \frac{\tau_{\text{fail}}}{\tau_{\text{allow}}}$$

- In all the equations, F.S. is chosen to be greater than 1, to avoid potential for failure
- Specific values will depend on types of material used and its intended purpose

Factor of safety & allowable stress

- To determine area or dimensions of section subjected to a *normal force*, use

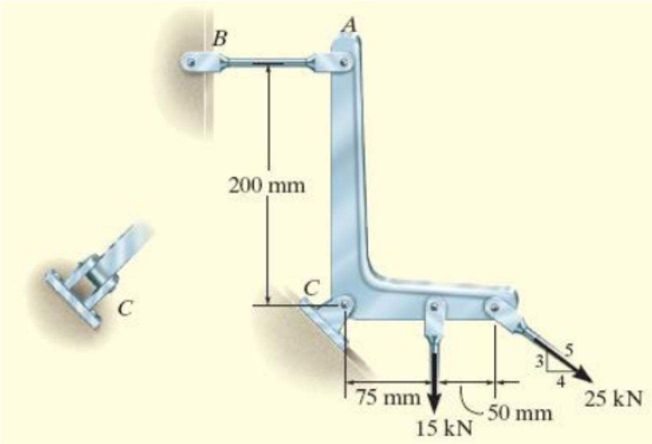
$$A = \frac{P}{\sigma_{allow}}$$

- To determine area or dimensions of section subjected to a *shear force*, use

$$A = \frac{V}{\tau_{\text{allow}}}$$

Example

The control arm is subjected to the loading. Determine to the nearest 5 mm the required diameter of the steel pin at C if the allowable shear stress for the steel is $\tau_{allowable} = 55$ MPa. Note in the figure that the pin is subjected to double shear.



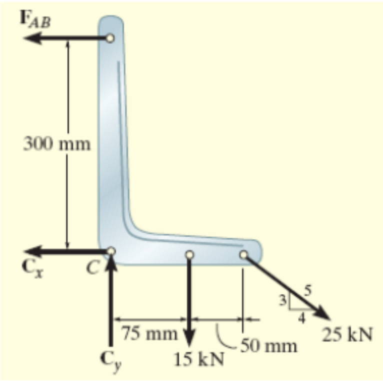
Solution:

For equilibrium we have

$$\circlearrowleft + \sum M_C = 0; \quad F_{AB}(0.2) = 15(0.075) - 25\left(\tfrac{3}{5}\right)(0.125) = 0 \Rightarrow F_{AB} = 15\text{ kN}$$

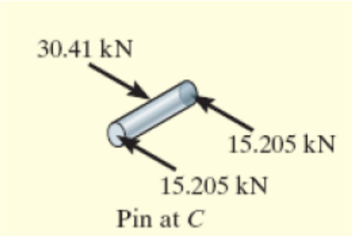
$$\rightarrow + \sum F_x = 0; \quad -15 - C_x + 25\left(\tfrac{4}{5}\right) = 0 \Rightarrow C_x = 5\text{ kN}$$

$$\uparrow + \sum F_y = 0; \quad C_y - 15 - 25\left(\tfrac{3}{5}\right) = 0 \Rightarrow C_y = 30\text{ kN}$$



Example

Solution:



The pin at C resists the resultant force at C. Therefore,

$$F_C = \sqrt{(5)^2 + (30)^2} = 30.41\,\text{kN}$$

The pin is subjected to double shear, a shear force of 15.205 kN acts over its cross-sectional area *between* the arm and each supporting leaf for the pin.

The required area is

$$A = \frac{V}{\tau_{allowable}} = \frac{15.205}{55 \times 10^3} = 276.45 \times 10^{-6}\ \text{m}^2$$

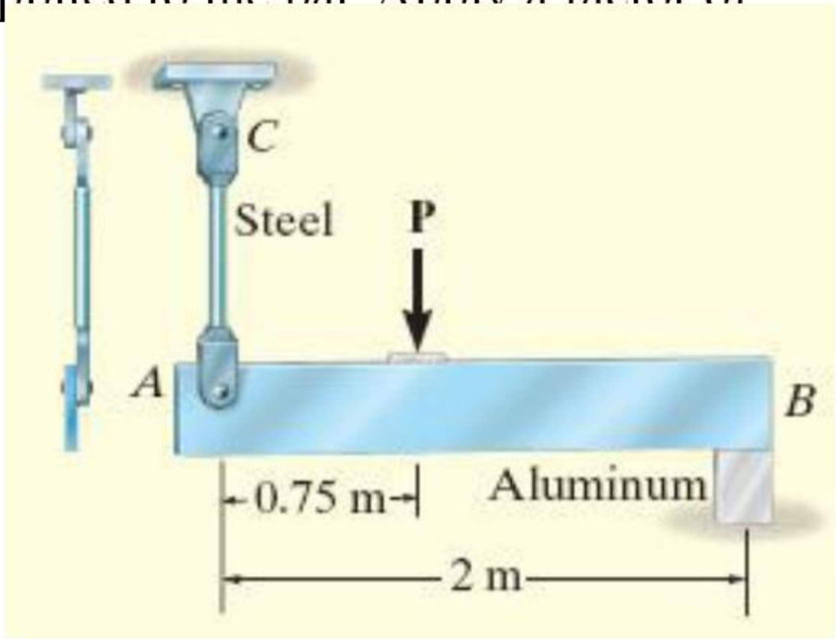
$$\pi\left(\frac{d}{2}\right)^2 = 246.45\ \text{mm}^2$$

$$d = 18.8\ \text{mm}$$

Use a pin with a diameter of d = 20 mm. (Ans)

Example

The rigid bar *AB* supported by a steel rod *AC* having a diameter of 20 mm and an aluminum block having a cross sectional area of 1800 mm^2. The 18-mm-diameter pins at *A* and *C* are subjected to *single shear*. If the failure stress for the steel and aluminum is $(\sigma_{st})_{fail} = 680\,\text{MPa}$ and $(\sigma_{al})_{fail} = 70\,\text{MPa}$ respectively, and the failure shear stress for each pin is $\tau_{fail} = 900\ \text{MPa}$ determine the largest load *P* that can be applied to the bar. Apply a factor of safety of F.S. = 2.



Solution:

The allowable stresses are

$$(\sigma_{st})_{allow} = \frac{(\sigma_{st})_{fail}}{F.S.} = \frac{680}{2} = 340\ \text{MPa}$$

$$(\sigma_{al})_{allow} = \frac{(\sigma_{al})_{fail}}{F.S.} = \frac{70}{2} = 35\ \text{MPa}$$

$$\tau_{allow} = \frac{\tau_{fail}}{F.S.} = \frac{900}{2} = 450\ \text{MPa}$$

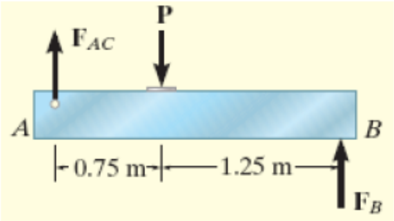
Example

Solution:

There are three unknowns and we apply the equations of equilibrium,

$$\circlearrowleft + \sum M_B = 0; \quad P(1.25) - F_{AC}(2) = 0 \qquad (1)$$

$$\circlearrowleft + \sum M_A = 0; \quad F_B(2) - P(0.75) = 0 \qquad (2)$$



We will now determine each value of P that creates the allowable stress in the rod, block, and pins, respectively.

For rod AC, $F_{AC} = (\sigma_{st})_{allow}(A_{AC}) = 340(10^6)\left[\pi(0.01)^2\right] = 106.8\text{ kN}$

Using Eq. 1 $P = \dfrac{(106.8)(2)}{1.25} = 171\text{ kN}$

For block B $F_B = (\sigma_{al})_{allow} A_B = 35(10^6)\left[1800(10^{-6})\right] = 63.0\text{ kN}$

Using Eq. 2, $P = \dfrac{(63.0)(2)}{0.75} = 168\text{ kN}$

Example

Solution:

For pin A or C, $V = F_{AC} = \tau_{allow} A = 450\left(10^6\right)\left[\pi(0.009)^2\right] = 114.5\,\text{kN}$

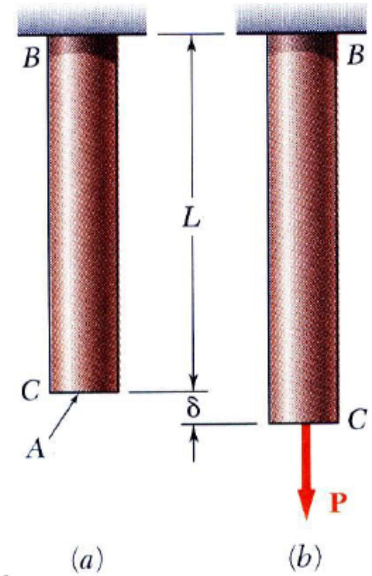
Using Eq. 1, $P = \dfrac{(114.5)(2)}{1.25} = 183\,\text{kN}$

When P reaches its *smallest value* (168 kN), it develops the allowable normal stress in the aluminium block. Hence,

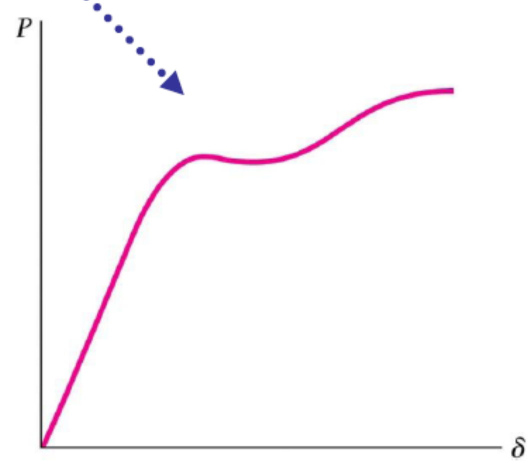
$$P = 168\,\text{kN (Ans)}$$

Normal Strain Under Axial Loading

- A rod *BC*, of length *L* and uniform cross-sectional area *A*, is suspended from *B*.
- A load ***P*** is applied to end *C*, then the rod elongates by δ.



- Plotting the magnitude *P* of the load against the deformation δ, the shown *load-deformation* diagram is obtained.
- **However**, this diagram *cannot* be used directly to predict the deformation of a rod of the same material but of different dimensions. It is logic that δ depends also on *cross-sectional area A* and *length L* of the rod under investigation.

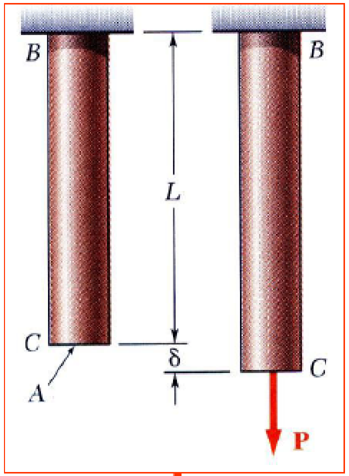


Normal Strain Under Axial Loading

- Define the ***normal strain*** **(ε)** in a rod under axial loading as: the *deformation per unit length of the rod*:

$$\varepsilon = \frac{\delta}{L}$$

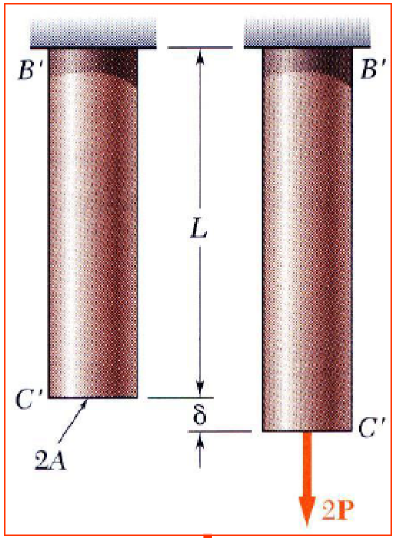
P, A, L



$$\sigma = \frac{P}{A}$$

$$\varepsilon = \frac{\delta}{L}$$

2P, 2A, L

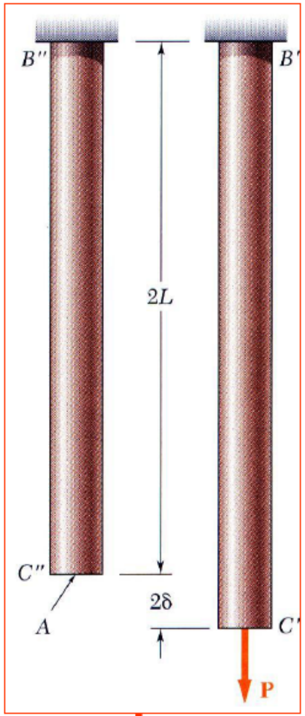


$$\sigma = \frac{2P}{2A} = \frac{P}{A}$$

$$\varepsilon = \frac{\delta}{L}$$

Normal Strain Under Axial Loading

$P, A, 2L$



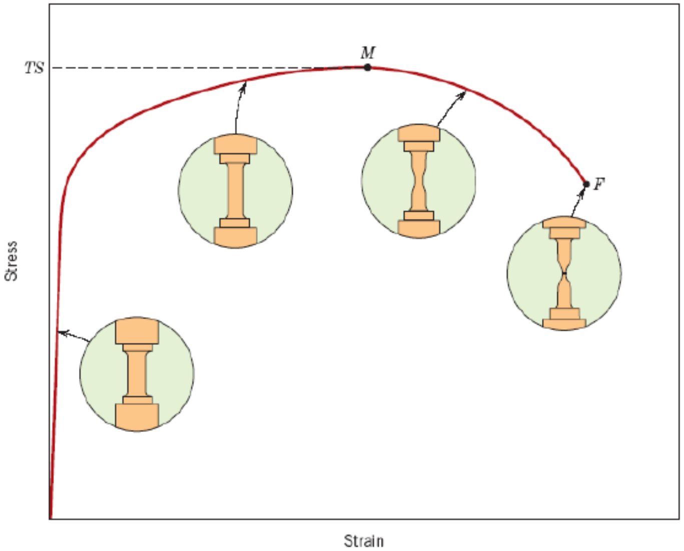
$$\sigma = \frac{P}{A}$$

$$\varepsilon = \frac{2\delta}{2L} = \frac{\delta}{L}$$

Finally, it is better to plot σ-ε relation instead of P-δ relation.

This diagram is called ***Stress-Strain*** *Diagram* and it can be plotted using the **tensile test machine**.

$$\sigma = \frac{P}{A}$$



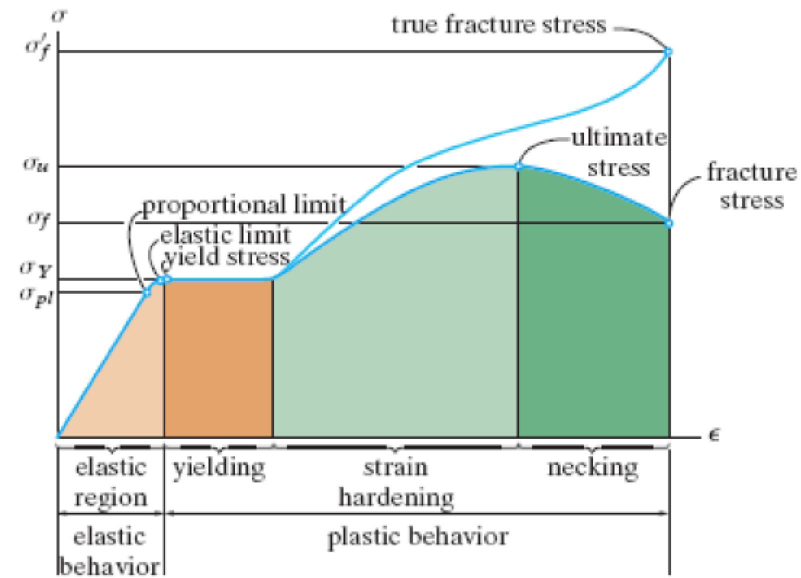
L is the initial length

$$\varepsilon = \frac{\delta}{L}$$

The Stress–Strain Diagram

- Elastic Behaviour
  - Stress is *proportional* to the strain.
  - Material is said to be *linearly elastic.*

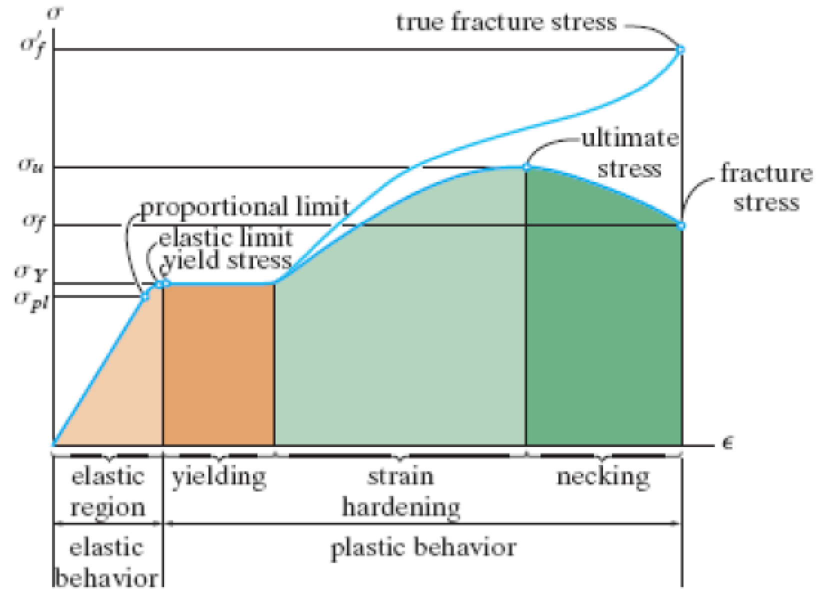
- Yielding
  - Increase in stress above elastic limit will cause material to *deform permanently.*



Conventional and true stress-strain diagrams for ductile material (steel) (not to scale)

The Stress–Strain Diagram

- Strain Hardening.
  - After yielding a further load will reaches a **ultimate stress**.
- *Necking*
  - At ultimate stress, cross-sectional area begins to decrease in a *localized* region of the specimen.
  - Specimen breaks at the ***fracture stress***.



Conventional and true stress-strain diagrams for ductile material (steel) (not to scale)

Hooke's law

- *Hooke's Law* defines the *linear relationship* between stress and strain within the elastic region.

$$\sigma = E\varepsilon$$

σ = stress

E = ***modulus of elasticity*** or ***Young's modulus***

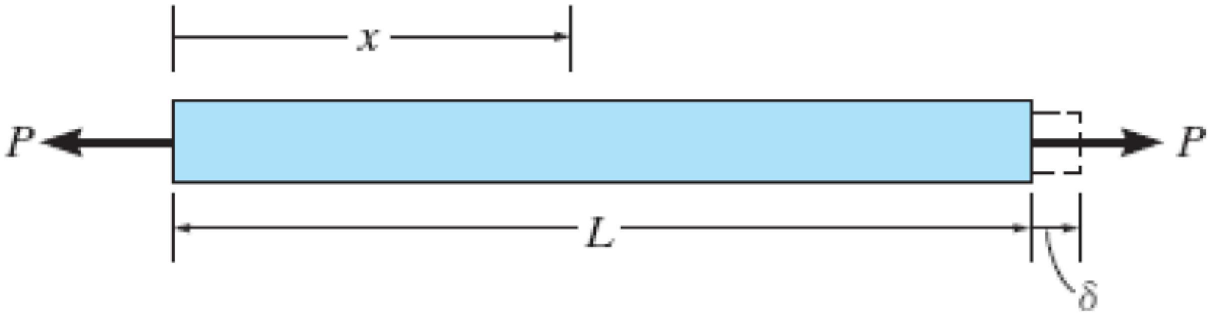
ε = strain

- *E* can be used only if a material has *linear–elastic* behaviour.

Axial deflection

Constant Load and Cross-Sectional Area

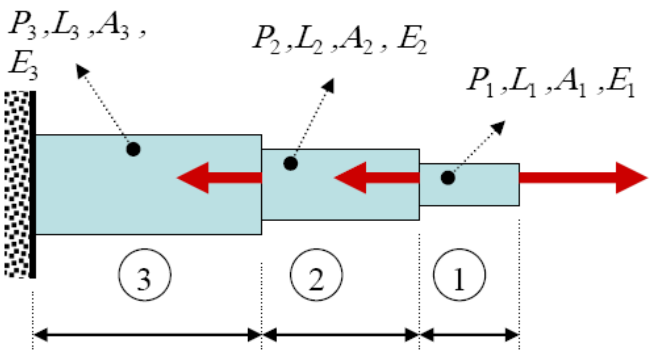
- When a constant external force is applied at each end of the member,



$$\delta = \frac{PL}{AE}$$

Axial deflection

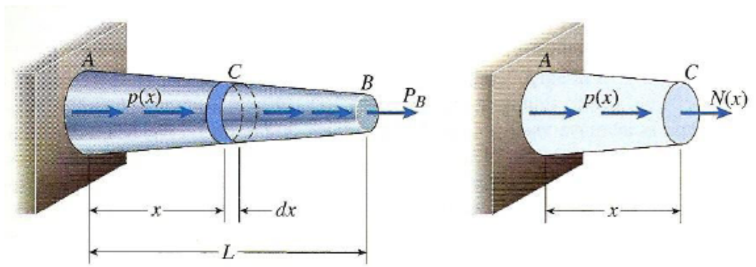
- *Bars loaded at several points and /or consist of several parts of various cross sections and possibly of different materials.*



*The deformation δ (e.g, at free end) is the **sum** of the deformations of its component parts*

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i}$$

- *Homogeneous Bar with variable cross section*

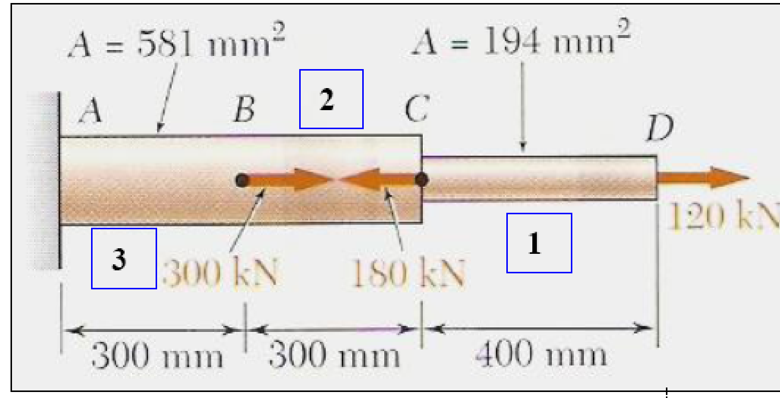


$$\delta = \int_0^L \frac{P\,dx}{EA(x)}$$

Note: If P and/or E are variables with x, then their variation must be considered.

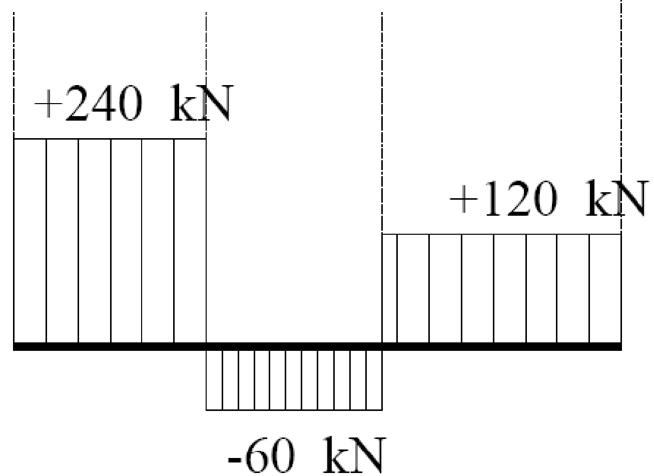
Example

Determine the deformation, at *D*, of the steel rod shown under the given loads ($E = 200$ GPa).



Solution:

Divide the rod into three parts:



Part 1:

$$P_1 = 120 \times 10^3 \;\; N$$
$$A_1 = 194 \;\; mm^2$$
$$L_1 = 400 \;\; mm$$

Part 2:

$$P_2 = -60 \times 10^3 \;\; N$$
$$A_2 = 581 \;\; mm^2$$
$$L_2 = 300 \;\; mm$$

Part 3:

$$P_3 = 240 \times 10^3 \;\; N$$
$$A_3 = 581 \;\; mm^2$$
$$L_3 = 300 \;\; mm$$

Example

$$\delta = \sum_i \frac{P_i L_i}{A_i E_i} = \frac{1}{E}\left(\frac{P_1 L_1}{A_1} + \frac{P_2 L_2}{A_2} + \frac{P_3 L_3}{A_3}\right)$$

$$= \frac{1}{200\times 10^3}\left(\frac{120\times 10^3 \times 400}{194} - \frac{60\times 10^3 \times 300}{581} + \frac{240\times 10^3 \times 300}{581}\right)$$

$$= 1.702 \quad mm$$

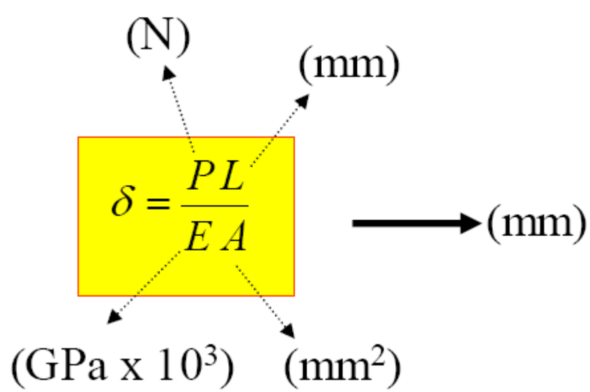
Note:

$$\delta = \delta_D = \frac{P_1 L_1}{E_1 A_1} + \frac{P_2 L_2}{E_2 A_2} + \frac{P_3 L_3}{E_3 A_3}$$

$$= \delta_{D/C} + \delta_{C/B} + \delta_{B/A}$$

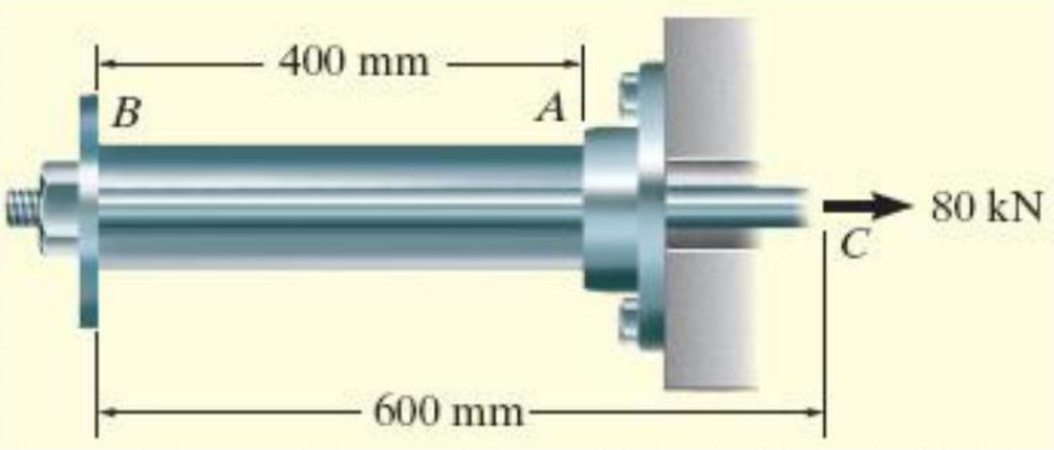
$$= \delta_{D/A}$$

Note: if you want !!



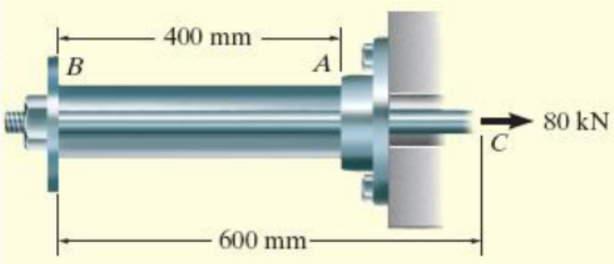
Example

The assembly consists of an aluminum tube *AB* having a cross-sectional area of 400 mm². A steel rod having a diameter of 10 mm is attached to a rigid collar and passes through the tube. If a tensile load of 80 kN is applied to the rod, determine the displacement of the end *C* of the rod. (E_{st} = 200 GPa, E_{al} = 70 GPa)



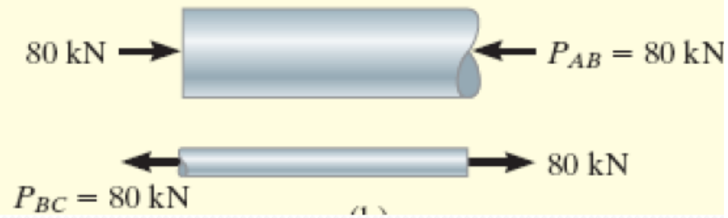
Example

Solution:



Find the displacement of end C with respect to end B.

$$\delta_{C/B} = \frac{PL}{AE} = \frac{\left[+80\left(10^3\right)\right](0.6)}{\pi(0.005)\left[200\left(10^9\right)\right]} = +0.003056\,\text{m} \rightarrow$$



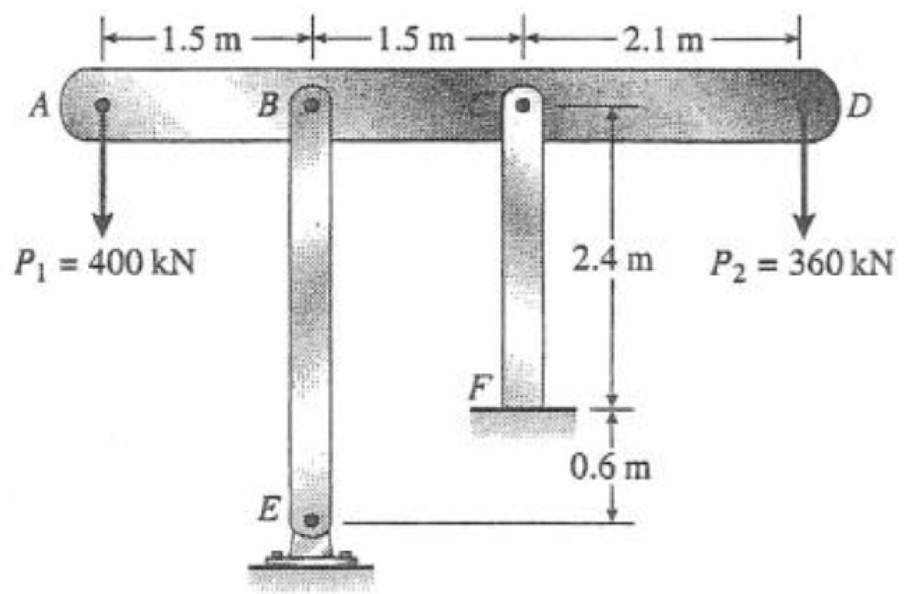
Displacement of end B with respect to the *fixed* end A,

$$\delta_B = \frac{PL}{AE} = \frac{\left[-80\left(10^3\right)\right](0.4)}{\left[400\left(10^{-6}\right)\right]\left[70\left(10^9\right)\right]} = -0.001143 = 0.001143\,\text{m} \rightarrow$$

Since both displacements are to the right, $\delta_C = \delta_B + \delta_{C/B} = 0.0042\,\text{m} = 4.20\,\text{mm} \rightarrow$

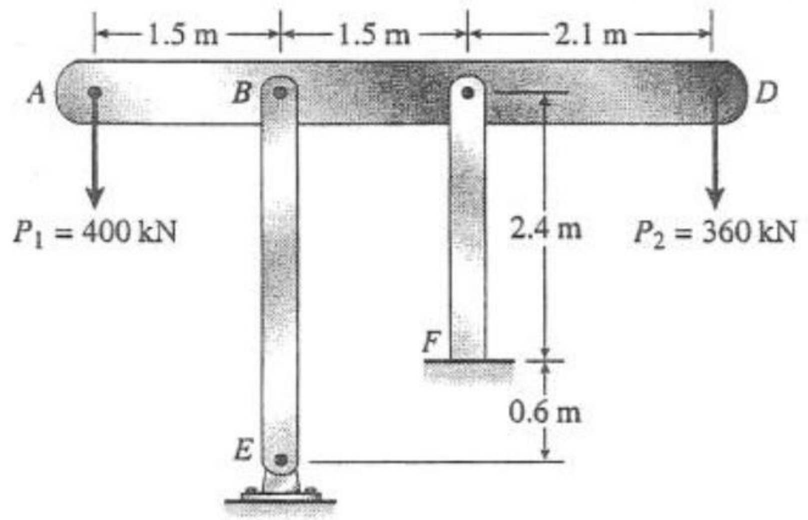
Example

The horizontal rigid beam *ABCD is supported* by vertical bars *BE and CF and is loaded by vertical forces* $P_1 = 400$ *kN and* $P_2 = 360$ *kN acting at points A and D,* respectively (see figure). Bars *BE and CF are made of steel* (*E 200 GPa) and have cross-sectional areas* $A_{BE} = 11{,}100\ mm^2$ and $A_{CF} = 9{,}280\ mm^2$. *The distances between various points on* the bars are shown in the figure. Determine the vertical displacements δ_A *and* δ_D *of points A* and *D, respectively.*



Example

solution



$A_{BE} = 11{,}100\ \text{mm}^2$

$A_{CF} = 9{,}280\ \text{mm}^2$

$E = 200\ \text{GPa}$

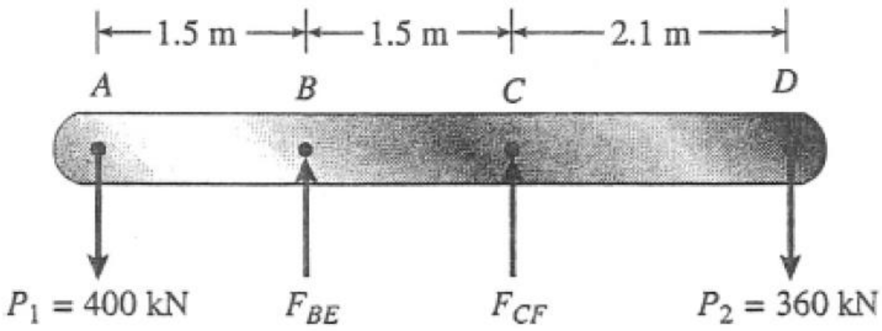
$L_{BE} = 3.0\ \text{m}$

$L_{CF} = 2.4\ \text{m}$

$P_1 = 400\ \text{kN};\ P_2 = 360\ \text{kN}$

Example

FREE-BODY DIAGRAM OF BAR *ABCD*



$$\Sigma M_B = 0$$

$$(400\text{ kN})(1.5\text{ m}) + F_{CF}(1.5\text{ m}) - (360\text{ kN})(3.6\text{ m}) = 0$$

$$F_{CF} = 464\text{ kN}$$

$$\Sigma M_C = 0$$

$$(400\text{ kN})(3.0\text{ m}) - F_{BE}(1.5\text{ m}) - (360\text{ kN})(2.1\text{ m}) = 0$$

$$F_{BE} = 296\text{ kN}$$

SHORTENING OF BAR *BE*

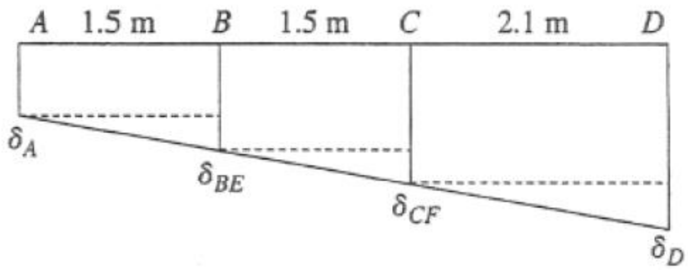
$$\delta_{BE} = \frac{F_{BE}L_{BE}}{EA_{BE}} = \frac{(296\text{ kN})(3.0\text{ m})}{(200\text{ GPa})(11{,}100\text{ mm}^2)} = 0.400\text{ mm}$$

Example

SHORTENING OF BAR CF

$$\delta_{CF} = \frac{F_{CF}L_{CF}}{EA_{CF}} = \frac{(464\text{ kN})(2.4\text{ m})}{(200\text{ GPa})(9{,}280\text{ mm}^2)}$$
$$= 0.600\text{ mm}$$

DISPLACEMENT DIAGRAM



$$\delta_{BE} - \delta_A = \delta_{CF} - \delta_{BE} \text{ or } \delta_A = 2\delta_{BE} - \delta_{CF}$$
$$\delta_A = 2(0.400\text{ mm}) - 0.600\text{ m}$$
$$= 0.200\text{ mm} \leftarrow$$

(Downward)

$$\delta_D - \delta_{CF} = \frac{2.1}{1.5}(\delta_{CF} - \delta_{BE})$$

or $$\delta_D = \frac{12}{5}\delta_{CF} - \frac{7}{5}\delta_{BE}$$
$$= \frac{12}{5}(0.600\text{ mm}) - \frac{7}{5}(0.400\text{ mm})$$
$$= 0.880\text{ mm} \leftarrow$$

(Downward)

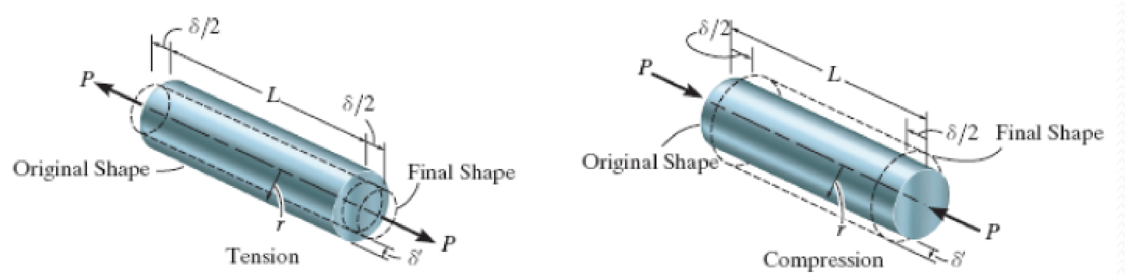
Poisson's ratio

- ***Poisson's ratio***, *ν* (nu), states that in the *elastic range,* the *ratio* of these strains is a *constant* since the deformations are proportional.

$$\nu = -\frac{\varepsilon_{lat}}{\varepsilon_{long}}$$

Poisson's ratio is *dimensionless*.
Typical values are 1/3 or 1/4.

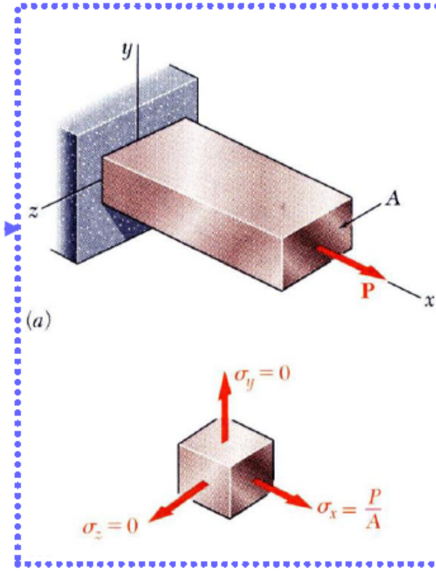
- Negative sign since *longitudinal elongation* (positive strain) causes *lateral contraction* (negative strain), and vice versa.



Poisson's ratio

- For a slender bar subjected to axial loading:

$$\varepsilon_x = \frac{\sigma_x}{E} \qquad \sigma_y = \sigma_z = 0$$



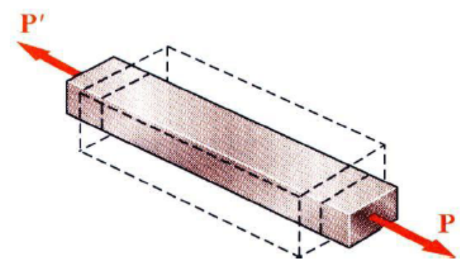
- The elongation in the *x*-direction is accompanied by a ***contraction*** in the other lateral directions. Assuming that the *material is isotropic* (no directional dependence),

$$\varepsilon_y = \varepsilon_z \neq 0$$

- Poisson's ratio is defined as

$$\nu = -\frac{\textit{lateral strain}}{\textit{axial strain}} = -\frac{\varepsilon_y}{\varepsilon_x} = -\frac{\varepsilon_z}{\varepsilon_x}$$

Note that ν is a property of the material.



- So; for a ***homogeneous isotropic*** material subjected to an ***axial loading in x*** direction:

$$\varepsilon_x = \frac{\sigma_x}{E} \quad \textit{and} \quad \varepsilon_y = \varepsilon_z = -\frac{\nu\sigma_x}{E}$$

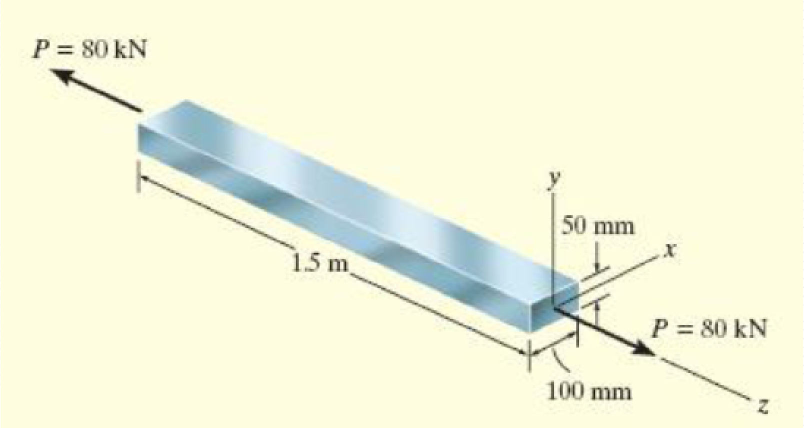
Example

A bar made of A-36 steel has the dimensions shown. If an axial force of is applied to the bar, determine the change in its length and the change in the dimensions of its cross section after applying the load. The material behaves elastically.

Solution:

The normal stress in the bar is

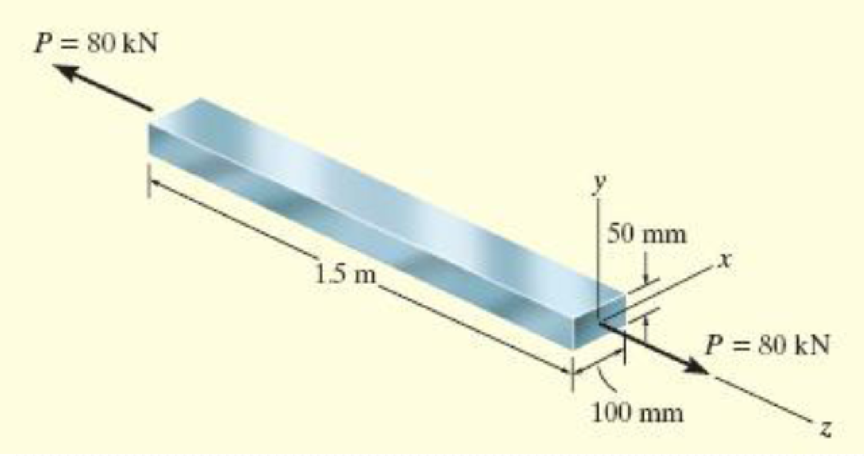
$$\sigma_z = \frac{P}{A} = \frac{80(10^3)}{(0.1)(0.05)} = 16.0(10^6)\,\text{Pa}$$



From the table for A-36 steel, E_{st} = 200 GPa

$$\varepsilon_z = \frac{\sigma_z}{E_{st}} = \frac{16.0(10^6)}{200(10^6)} = 80(10^{-6})\,\text{mm/mm}$$

Example



Solution:

The axial elongation of the bar is therefore

$$\delta_z = \varepsilon_z L_z = \left[80\left(10^{-6}\right)(1.5)\right] = 120\,\mu\text{m} \quad \text{(Ans)}$$

The contraction strains in *both* the *x* and *y* directions are

$$\varepsilon_x = \varepsilon_y = -\nu_{st}\varepsilon_z = -0.32\left[80\left(10^{-6}\right)\right] = -25.6\ \mu\text{m/m}$$

The changes in the dimensions of the cross section are

$$\delta_x = \varepsilon_x L_x = -\left[25.6\left(10^{-6}\right)(0.1)\right] = -2.56\,\mu\text{m} \quad \text{(Ans)}$$

$$\delta_y = \varepsilon_y L_y = -\left[25.6\left(10^{-6}\right)(0.05)\right] = -1.28\,\mu\text{m} \quad \text{(Ans)}$$